Advanced Time Series Econometrics

Autumn 2023

Exercises 1: Probability Solutions By Theo

1. F_1 is a σ-field since $\emptyset$ and Ω belong to F_1. F_1 is the smallest σ-field on Ω. F_2 is a σ-field since $\emptyset$, Ω, A and A^c belong to F_2. F_3 is a σ-field since all subsets of Ω (countable) belong to F_3. F_3 is the largest σ-field on Ω.

2. Using the definition of a σ-field, the sets $\emptyset$ and Ω belong to the σ-field $\sigma(A_1)$, which proves that $F_1 = \sigma(A_1)$. Similarly, the sets $\emptyset$, Ω, A and A^c belong to the σ-field $\sigma(A_2)$, which proves that $F_2 = \sigma(A_2)$. The set F_3 is a σ-field, and by tautology $F_3 = \sigma(\{F_3\})$.

3. (a) Take $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. Then

$$\bigcup_{n=1}^{\infty} \left[x_1 + \frac{\epsilon}{n}, x_2 - \frac{\epsilon}{n}\right] = (x_1, x_2);$$

   for $\epsilon < \frac{1}{2(x_2 - x_1)}$. Countable unions of closed intervals are open intervals. The Borel σ-field $\mathcal{B}$ therefore contains all open intervals.

   (b) The Borel σ-field $\mathcal{B}$ is defined to be generated by the half lines $(-\infty, x]$. Since

$$\bigcup_{n=1}^{\infty} (x - n, x] = (-\infty, x];$$

   the sets $(-\infty, x]$ can be generated by countable unions of finite intervals. Hence, they generate the same Borel σ-field.

4. Let $A_1, A_2, \ldots$ be a sequence of sets such that the collection $\{A_n : n \in \mathbb{R}\}$ is a partition of $\Omega = \{1, 2, \ldots\}$ and $P(A_1) = P(A_2) = \ldots = p > 0$. Then

$$\sum_{i=1}^{\infty} P(A_i) = \infty,$$

   Which is a contradiction of countable additivity. On the other hand, if $p = 0$, then the probability axiom $P(\Omega) = 1$ is not satisfied.

5. (a) If $a > 0$, then $\{\omega : aX(\omega) \le x\} = \{\omega : X(\omega) \le \frac{x}{a}\} \in \mathcal{F}$, since X is a random variable. If $a = 0$,

$$\{\omega : aX(\omega) \le x\} = \begin{cases} \emptyset, & \text{if } x < 0; \\ \Omega, & \text{if } x \ge 0; \end{cases}$$

In either case, the event belongs to $\mathcal{F}$. If $a < 0$,

$$\{\omega : aX(\omega) \le x\} = \{\omega : X(\omega) \ge \frac{x}{a}\} = \left(\bigcup_{n=1}^{\infty} \left\{ \omega : X(\omega) \le \frac{x}{a} - \frac{1}{n} \right\} \right)^c,$$

Which belongs to $\mathcal{F}$, since it is the complement of a countable union of events belonging to $\mathcal{F}$. The second equality follows from the first of De Morgan's laws. Linear functions are continuous. The conclusion that aX is a random variable follows from Theorem 8 (Theorem 13.2 of Billingsley 1995).

(b) For $\omega \in \Omega$, $X(\omega) - X(\omega) = 0$, so it follows from (a) with $a = 0$ that the zero random variable is a random variable. Similarly, $X + X = 2X$, and it follows from (a) with $a = 2$ that $2X$ is a random variable.

6. The limit, if it exists, is $X = 0$.

(a) By Lemma 10, X_n converges to X almost surely, as $n \to \infty$, if for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty.$$

We have

$$\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) = \sum_{n=1}^{\infty} P(X_n = n) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < \infty,$$

if $\alpha > 1$. Therefore, X_n converges to 0 almost surely, as $n \to \infty$, if $\alpha > 1$.

(b) Convergence in probability follows from

$$P(|X_n| > \varepsilon) = P(X_n = n) = \frac{1}{n^{\alpha}} \to 0,$$

as $n \to \infty$. Therefore, X_n converges to 0 almost surely, as $n \to \infty$, for all $\alpha > 0$.

(c) For convergence in quadratic mean,

$$E(X_n^2) = 0^2 \cdot \left(1 - \frac{1}{n^{\alpha}}\right) + n^2 \cdot \frac{1}{n^{\alpha}} = \frac{1}{n^{\alpha - 2}} \to 0,$$

as $n \to \infty$ for $\alpha > 2$. Therefore, X_n converges to 0 almost surely, as $n \to \infty$, if $\alpha > 2$.

Remarks: Convergence in probability follows from almost sure convergence under the additional assumption that $\alpha > 1$. Convergence in probability follows from convergence in quadratic mean under the additional assumption that $\alpha > 2$.

7.

(a) $A \in \mathcal{F}_{11}$, but $A \notin \mathcal{F}_{10}$. The smallest n is $n = 11$.

(b) $A \notin \mathcal{F}_n$ for any n. There is no smallest n such that $A \in \mathcal{F}_n$.

(c) $A \in \mathcal{F}_{100}$, but $A \notin \mathcal{F}_{99}$. The smallest n is $n = 100$.

(d) Since $A = \emptyset$, $A \in \mathcal{F}_n$ for all $n = 1, 2, \ldots$. The smallest n is $n = 1$.

8 (a) Verify (i)–(iii) in Definition 23.

(i) $E(|X_n g_{n-1}|) < \infty$ is assumed.

(ii) $\{X_n g_{n-1}\}$ is adapted to F_n.

(iii) $E(X_n g_{n-1} \mid F_{n-1}) = g_{n-1} E(X_n \mid F_{n-1}) = 0$. The first equality follows since g_{n-1} is adapted to F_{n-1} ('taking out what is known') and the second since $\{X_n g\}$ is a martingale difference sequence with respect to F_n.

(b) By the law of iterated expectations,

$$E(X_n) = E(E(X_n \mid F_{n-1})) = 0$$

and

$$E(X_n g_{n-1}) = E(E(X_n g_{n-1} \mid F_{n-1})) = E(g_{n-1} E(X_n \mid F_{n-1})) = 0.$$

Hence,

$$\mathrm{Cov}(X_n, g_{n-1}) = E(X_n g_{n-1}) - E(X_n)E(g_{n-1}) = 0.$$

(c) Put $g_{n-1} = X_{t-h}$. The result in (b) implies

$$\mathrm{Cov}(X_n, X_{n-h}) = 0; \quad h > 0.$$